

elements of the S matrix could then be expressed in some simple manner in terms of these eigenstates. The main problem facing the theory would then be shifted to that of calculating these eigenstates, where one would have a much better chance of separating out the infinities which arise in the present theory.

ACKNOWLEDGMENT

The author would like to express his thanks to Dr. F. Rohrlich, Dr. Y. Nambu, and Dr. P. Bergmann for their many interesting and helpful comments in connection with this work.

PHYSICAL REVIEW

VOLUME 94, NUMBER 3

MAY 1, 1954

Low-Energy Limits and Renormalization in Meson Theory

S. DESER* AND W. E. THIRRING,† *Institute for Advanced Study, Princeton, New Jersey,*

AND

M. L. GOLDBERGER,‡ *Princeton University, Princeton, New Jersey*

(Received January 7, 1954)

A technique is developed for deriving rigorous expressions for zero-energy mesonic processes. Making use of the ambiguity of mesonic charge renormalization, the coupling constant is defined by zero-energy pion-nucleon scattering. The threshold photomeson production amplitude is also calculated. The experimental value of the coupling constant defined by scattering is at least an order of magnitude less than that of the coupling constant deduced from the photoproduction data. It is shown that pair suppression effects are absent in both cases. The possibility that the results obtained imply disagreement of pseudoscalar theory with experiment is discussed.

I. INTRODUCTION

IT is almost universally accepted at present that π mesons are pseudoscalar, and that to a very good approximation isotopic spin is a good quantum number in mesonic-nucleon processes. Unfortunately, it is not at all clear to what extent the conventional pseudoscalar symmetrical meson theory correctly describes such processes. Because of the difficulty of carrying out accurate calculations, it has been hard to distinguish between the predictions of the theory and the effects of the particular approximation schemes hitherto employed. In order to avoid extraneous difficulties, it seems advisable to concentrate on understanding low-energy phenomena, where both nucleons and mesons are nonrelativistic. Two such processes, which will be discussed in the present paper, are threshold photomeson production and meson-nucleon scattering at zero energy.

By the use of special techniques it has been possible to evaluate the matrix elements for these two processes rigorously in the limit of vanishing meson mass. These are, of course, purely formal results; however, as is discussed at length below, one of them, say the meson scattering, may be used to assign a precise numerical value to the coupling constant and the other then in principle serves as a check on the agreement between theory and experiment. Unfortunately we are not able

to evaluate the photo-meson matrix element explicitly, and thus unambiguously settle this crucial question; nor is the present experimental data sufficiently accurate to give a reliable number for the coupling constant. We can, however, say the following: If in the computation of meson-nucleon scattering one may without serious error set the mass of the real mesons equal to zero in the portion of the scattering amplitude which is independent of isotopic spin, one deduces from the existing data the value $g^2/4\pi \sim \frac{1}{3}$. If one may assume that with such a small value of the coupling constant perturbation theory is valid (at least in an asymptotic sense), then one may conclude that the present theory does not correctly describe the behavior of π mesons. Of course, should the value of $g^2/4\pi$ turn out to be much larger or should the neglect of the meson mass mentioned above prove unwarranted, we can make no definite statement.

Before the theory can be compared with experiment, it must, of course, be renormalized. Pseudoscalar theory (with pseudoscalar coupling) is a "renormalizable" theory in the conventional sense of the term; however, in contrast to quantum electrodynamics, the renormalization program may here be carried out in a variety of nonequivalent ways. This fact is of great importance in any attempts to ascertain the physical content of mesonic calculations. Charge renormalization is uniquely defined in electrodynamics;¹ this is closely related to the existence of a charge conservation law and finds its

* Parker Fellow, Harvard University.

† On leave of absence from Physikalisches Institut, Universität Bern, Bern, Switzerland.

‡ On leave of absence from the University of Chicago, Chicago, Illinois.

¹ G. Källén (to be published).

formal expression in the well known Ward identity.² One can define the charge either from the requirement that Coulomb's law should hold for large distances, or that the motion of a low-energy electron in a slowly varying field be describable classically, or that low-frequency Compton scattering be given by the Thomson formula. It turns out that all of these definitions lead to the same value for e . The underlying physical reason is that a slowly varying electromagnetic field is insensitive to the shape of the charge distribution, and radiative corrections cannot change the total charge. The above-mentioned experiments are characterized by interactions over macroscopic distances or by processes in which there is negligible energy-momentum transfer to the charged particle. In meson theory the finite meson rest mass precludes the first type of definition (as it implies a finite range of forces) and introduces a difference between processes in which the four-momentum transfer to a nucleon is zero (such as forward scattering of mesons by nucleons), or those in which the transfer is of the order of the meson rest mass (such as threshold photomeson production, or threshold production in a nucleon-nucleon collision). The ambiguity introduced by the necessity of choosing one of the two types of processes as a basis for the definition of the coupling constant g has been stressed by Thellung.³ As we shall see, however, even in the limit of vanishing meson mass, the two alternatives lead to different definitions of the renormalized coupling constant. There is another (physically less important) ambiguity introduced by the absence of a conservation law for mesonic charge which will be discussed in Sec. II.

By analogy with electrodynamics, one would like to define the mesonic charge by one of the zero-energy phenomena discussed above, choosing the renormalized coupling constant in such a way that the rigorous matrix element becomes identical with the result of lowest-order perturbation theory. In a recent paper,⁴ Kroll and Ruderman have shown how this may be done for the process of threshold photoproduction of charged mesons in the limit of vanishing meson mass. To carry out this program, they give very explicit instructions as to how the various finite functions of the theory are to be obtained, and any computations based on their coupling constant must be performed in accordance with their rules. As we have mentioned, such a procedure is only one of many possible and one of the purposes of the present paper is to discuss an alternative prescription.

A natural way to define g would appear to be (in analogy to Compton scattering) the scattering of zero-energy mesons by nucleons. This process has the virtue of involving only the nucleon and meson fields, and further is one in which the condition of zero four-

momentum transfer is actually met. More specifically, we shall base our definition of g on the scattering of zero kinetic energy meson in the limit of vanishing meson mass, and insist that the rigorous matrix element in this limit be just the second order perturbation theory result with renormalized coupling constant and nucleon mass.

In order to do more than merely state this requirement on a renormalization procedure, we must develop a method for computing the exact matrix element for meson scattering in the limit of vanishing meson mass. We will then know what the consequences of our prescription are, that is what functions are absorbed into the definition of g . In Sec. II such a technique is developed. It is based on the ideas of Schwinger⁵ and Feynman⁶ who have shown that if the motion of a particle in external fields is known one may compute all relevant quantities by means of appropriate variational derivatives with respect to these fields.⁷ In our case, since we are concerned with zero four-momentum transfer, we may specialize to a constant external field (which cannot transfer momentum) and thus replace the relevant variational derivatives by ordinary derivatives. A detailed discussion of the renormalization of the nucleon propagation function is given and a new method of renormalizing the meson vertex operator Γ_5 is presented. In Sec. III, an exact computation of meson nucleon scattering is carried out and the renormalized coupling constant is introduced. The appropriate generalization of our formalism to include electromagnetic effects is treated in Sec. IV, and is used to compute the threshold photoproduction of mesons. The well-known result⁸ of the vanishing of radiative corrections to Compton scattering for zero-frequency photons is also rederived using our formalism. The various formal considerations in Secs. II, III, and IV are illustrated in detail by explicit perturbation theory computations of radiative corrections with the different renormalization prescriptions.

In Sec. V, we compare our general results with experiment. The meson scattering amplitude derived in Sec. III cannot be directly compared with experiment, since, of course, the meson mass is not zero. It has recently been pointed out,⁹ however, that the general structure of the zero energy scattering amplitude is

$$R_{ji} = A\delta_{ji} + (\mu/M)[\tau_j, \tau_i]B, \quad (1.1)$$

where A and B are even functions of the meson mass which we assume are well behaved in the limit $\mu \rightarrow 0$. By our choice of mesonic charge renormalization we can compute A exactly in the limit of μ going to zero; hence, if we assume that the $(\mu/M)^2$ corrections to A

² J. C. Ward, Phys. Rev. **77**, 293 (1950); **78**, 182 (1950); **84**, 897 (1951).

³ A. Thellung, Helv. Phys. Acta. **25**, 307 (1952).

⁴ N. M. Kroll and M. A. Ruderman, Phys. Rev. **93**, 233 (1953).

⁵ J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452 (1951).

⁶ R. P. Feynman, Phys. Rev. **80**, 440 (1950).

⁷ The explicit formulation of the meson-nucleon problem in these terms has been given by S. Deser and P. Martin, Phys. Rev. **90**, 1075 (1953).

⁸ W. Thirring, Phil. Mag. **41**, 1193 (1950); See also reference 4.

⁹ M. Gell-Mann and M. L. Goldberger (to be published).

are negligible, knowing the S -wave scattering lengths, we can compute the coupling constant ($A(0) \sim g^2$). We do not assume that the masses of virtual mesons are zero. In fact, our considerations remain valid irrespective of the nature of the virtual particles, provided only that the various symmetry principles assumed are not violated by their presence. We find by this means coupling constants of at least an order of magnitude smaller than those previously considered in pseudoscalar theory. The significance of this result is discussed in relation to various suggested approximations to the theory and to its consistency with experiment. Finally, in the appendix, a technique is developed to compute μ/M corrections to the various functions in the theory and applied to obtain the form of the B in (1.1).

II. FORMAL THEORY

Consider the motion of a free nucleon in a constant symmetric pseudoscalar meson field, α_i . The Dirac equation (for a nucleon of four-momentum p) becomes

$$(i\gamma \cdot p + m + i\alpha \cdot \tau \gamma_5) \psi(p, \alpha) = 0, \quad (2.1)$$

with the energy-momentum relation $p^2 + m^2 + \alpha^2 = 0$. The corresponding propagation function, $S(p, \alpha)$, is defined by

$$S^{-1}(p, \alpha) = i\gamma \cdot p + m + i\alpha \cdot \tau \gamma_5. \quad (2.2)$$

We observe that

$$\frac{1}{i} \frac{\partial}{\partial \alpha_j} S = -S \tau_j \gamma_5 S, \quad (2.3)$$

and that consequently the interaction between a meson of zero four-momentum, charge index j , and a nucleon may be represented by differentiation of the propagation function with respect to α_j .

When the nucleon interacts with the quantized pseudoscalar meson field, $S(p, \alpha)$ is replaced by a modified propagation, $S'(p, \alpha)$. In view of Eq. (2.3), the effect of differentiating $S'(p, \alpha)$ with respect to α_j is equivalent to the interaction of the coupled nucleon with a zero four-momentum meson of charge j .¹⁰ Similarly, its interaction with n such mesons may be obtained from the n th derivative of $S'(p, \alpha)$. This is a special case of the general variational derivative technique developed by Schwinger and others^{5,6} in which variational derivatives may be replaced by ordinary derivatives.¹¹ Our method is somewhat analogous to use of the Ward trick¹² in quantum electrodynamics which has been extensively applied in connection with zero-frequency photon problems.

To proceed with our development we must exhibit the structure of the modified propagation function,

¹⁰ This may be seen by noting that differentiation of any graph composing inserts the meson vertex in all possible ways.

¹¹ If we were to set the external field α equal to zero after all differentiations, our results would coincide with usual pseudoscalar theory; we are not, however, forced to put α equal to zero and for the time being will not do so.

¹² Reference 2. Its statement is that $-i\partial/\partial p_\mu \cdot S'^{-1}(p) = \Gamma_\mu^1(p, p)$.

$S'(p, \alpha)$, mentioned above. From invariance requirements, we conclude that the most general form may be written as

$$S'^{-1}(p, \alpha) = i\gamma \cdot p g_0 + m f_0 + i\alpha \cdot \tau \gamma_5 d_0 + i\gamma \cdot p \alpha \cdot \tau \gamma_5 h_0, \quad (2.4)$$

where g_0, f_0, d_0 , and h_0 are scalar functions of $p^2, m^2, \alpha^2, \mu^2$ and, of course, the masses of any virtual fields which may be coupled to the meson-nucleon fields. Whereas all four terms may be present from the standpoint of invariance with respect to the full Lorentz group, we may actually eliminate h_0 by utilizing another symmetry principle: an examination of the general perturbation theoretic expression for $S'^{-1}(p, \alpha)$ shows that

$$[S'^{-1}(-p, -\alpha)]^T = C^{-1} [S'^{-1}(p, \alpha)] C, \quad (2.5)$$

where C is the charge conjugation matrix augmented to act also in isotopic spin space, with the properties

$$C^{-1} \gamma_\mu C = -\gamma_\mu^T, \quad C^{-1} \gamma_5 C = \gamma_5^T, \quad C^{-1} \tau C = -\tau^T. \quad (2.6)$$

(In the usual representation, $C = \gamma_4 \gamma_2 \tau_2$.) If we apply the requirement (2.5) to (2.4), we conclude that h_0 must be zero. Our propagation function is then¹³

$$S'^{-1}(p, \alpha) = i\gamma \cdot p g_0 + m f_0 + i\alpha \cdot \tau \gamma_5 d_0. \quad (2.7)$$

As a result of the interaction between the nucleon and the quantized meson field we know that a free nucleon satisfying the modified Dirac equation,

$$S'^{-1}(p, \alpha) \psi'(p, \alpha) = 0, \quad (2.8)$$

behaves as though it had an effective mass, different from m , and were moving in an external meson field different from α . Just as in electrodynamics it is convenient to introduce a mass renormalization so that a free, but "clothed," electron has the energy momentum relation $p^2 + M^2 = 0$, with M the experimental mass, we shall introduce a renormalization of M and α . Our demand is that a free, but clothed, nucleon must satisfy

$$p^2 + M^2 + \alpha'^2 = 0. \quad (2.9)$$

This will be possible if M and α' are suitably defined. If one iterates the modified Dirac equation, (2.8), one obtains

$$[p^2 g_0^2(p^2, \alpha^2) + m^2 f_0^2(p^2, \alpha^2) + \alpha^2 d_0^2(p^2, \alpha^2)] \psi'(p, \alpha) = 0. \quad (2.10)$$

We may evidently satisfy the requirement (2.9) by choosing M and α' such that

$$g_0(-M^2 - \alpha'^2, \alpha^2) = (m/M) f_0(-M^2 - \alpha'^2, \alpha^2) = (\alpha/\alpha') d_0(-M^2 - \alpha'^2, \alpha^2). \quad (2.11)$$

We assume that these equations may be solved for M and α' in terms of m and α , or rather, as is usually done, imagine that m and α are expressed in terms of the renormalized quantities M and α' . We shall in the future

¹³ It might be noted that while $\{i\gamma p, \alpha \cdot \tau \gamma_5\}$ does satisfy the charge conjugation requirements, it is identically zero.

also regard M as a function of α' . It is convenient to introduce new functions in place of g_0 , f_0 , and d_0 according to the definitions

$$g(p^2, \alpha'^2) = g_0(p^2, m^2, \alpha^2); \quad f(p^2, \alpha'^2) = (m/M)f_0(p^2, m^2, \alpha^2); \\ d(p^2, \alpha'^2) = (\alpha/\alpha')d_0(p^2, m^2, \alpha^2). \quad (2.12)$$

We have then

$$S'^{-1}(p, \alpha') = i\gamma \cdot p g(p^2, \alpha'^2) \\ + M f(p^2, \alpha'^2) + i\alpha' \cdot \tau \gamma_5 d(p^2, \alpha'^2), \quad (2.13)$$

and

$$g(-M^2 - \alpha'^2, \alpha'^2) = f(-M^2 - \alpha'^2, \alpha'^2) \\ = d(-M^2 - \alpha'^2, \alpha'^2) \equiv f. \quad (2.14)$$

Then for a free nucleon,

$$S'^{-1}(p, \alpha') = (i\gamma \cdot p + M + i\alpha' \cdot \tau \gamma_5) f. \quad (2.15)$$

Up to this point, we have been acting as though everything were finite in our theory. In practice, of course, the various functions g_0 , f_0 , and d_0 are divergent, and we must imagine a suitable cutoff to have been introduced so as to give meaning to our mass and α renormalizations. (It is worth pointing out that these particular renormalizations would be required even if the theory were completely convergent.) We wish now to extract the convergent part of our function S' just as is done in electrodynamics. We shall in fact follow the conventional procedure of Dyson¹⁴ and Ward² except that we retain our α field, and state the procedure in a somewhat different manner. We remark that for a bare nucleon one may define a particle current density, χ_μ , (not to be confused with the conventional electric current density $j_\mu = ie\bar{\psi}\gamma_\mu\psi$) by

$$\chi_\mu(p, \alpha) = \bar{\psi}(p, \alpha) \gamma_\mu \psi(p, \alpha) \\ = \bar{\psi}(p, \alpha) \left[\frac{1}{i} \frac{\partial}{\partial p_\mu} S^{-1}(p, \alpha) \right] \psi(p, \alpha). \quad (2.16)$$

The analogous quantity for a clothed nucleon is

$$\chi_\mu'(p, \alpha') = \bar{\psi}'(p, \alpha') \Gamma_\mu'(p, p, \alpha') \psi'(p, \alpha'), \quad (2.17)$$

where

$$\Gamma_\mu'(p, p, \alpha') \equiv \frac{1}{i} \frac{\partial}{\partial p_\mu} S'^{-1}(p, \alpha'). \quad (2.18)$$

In the limit as $\alpha' \rightarrow 0$, this becomes exactly the unrenormalized vertex operator of the usual theory (see Sec. IV). We now demand that after renormalization χ_μ' shall be numerically equal to the current density as computed from the renormalized spinors according to

$$\chi_\mu^{(e)} = \bar{\psi}_c(p, \alpha') \gamma_\mu \psi_c(p, \alpha'), \quad (2.19)$$

where ψ_c is a solution of

$$(i\gamma \cdot p + M + i\alpha' \cdot \tau \gamma_5) \psi_c(p, \alpha') = 0, \quad (2.20)$$

¹⁴ F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

and differs from ψ' by a constant factor. Physically, our requirement is that the particle current density of a one particle state is unmodified by interaction with the virtual field provided the appropriate adjustments of m and α are made. In the usual notation we write

$$\psi'(p, \alpha') = [Z_2(\alpha')]^{1/2} \psi_c(p, \alpha'), \\ \bar{\psi}'(p, \alpha') = [Z_2(\alpha')]^{1/2} \bar{\psi}_c(p, \alpha'), \quad (2.21) \\ S'(p, \alpha') = Z_2(\alpha') S_c(p, \alpha').$$

Our demand becomes

$$\bar{\psi}_c(p, \alpha') \Gamma_{\mu c}(p, p, \alpha') \psi_c(p, \alpha') \\ = \psi_c(p, \alpha') \gamma_\mu \psi_c(p, \alpha'), \quad (2.22)$$

where

$$\Gamma_{\mu c} = \frac{1}{i} \frac{\partial}{\partial p_\mu} S_c^{-1}(p, \alpha').$$

We have, carrying out the differentiation,

$$\Gamma_{\mu c} = Z_2(\alpha') [\gamma_\mu g - 2ip_\mu (i\gamma \cdot p g' + M f') \\ + 2\alpha' \cdot \tau \gamma_5 p_\mu d'], \quad (2.23)$$

where $g' = \partial g / \partial p^2$, etc. Using Eq. (2.20) and the elementary results deduced from it, namely,

$$\bar{\psi}_c(p, \alpha') \gamma_\mu \psi_c(p, \alpha') = (p_\mu / iM) \bar{\psi}_c(p, \alpha') \psi_c(p, \alpha'), \\ \bar{\psi}_c(p, \alpha') \tau_j \gamma_5 \psi_c(p, \alpha') = (\alpha_j' / iM) \bar{\psi}_c(p, \alpha') \psi_c(p, \alpha'), \quad (2.24)$$

one finds

$$\bar{\psi}_c(p, \alpha') \Gamma_{\mu c}(p, p, \alpha') \psi_c(p, \alpha') = Z_2(\alpha') [f + 2M^2(f' - g') \\ + 2\alpha'^2(d' - g')] \bar{\psi}_c(p, \alpha') \psi_c(p, \alpha'), \quad (2.25)$$

and all of the functions are to be evaluated at $p^2 = -M^2 - \alpha'^2$. Our requirement (2.22) fixes $Z_2(\alpha')$:

$$Z_2(\alpha') = [f + 2M^2(f' - g') + 2\alpha'^2(d' - g')]^{-1}. \quad (2.26)$$

In terms of new finite functions F , G , D , defined by

$$F = Z_2 f, \quad G = Z_2 g, \quad D = Z_2 d, \quad (2.27)$$

our convergent propagation function $S_c(p, \alpha')$ becomes

$$S_c^{-1}(p, \alpha') = i\gamma \cdot p G(p^2, \alpha'^2) + M F(p^2, \alpha'^2) \\ + i\alpha' \cdot \tau \gamma_5 D(p^2, \alpha'^2). \quad (2.28)$$

The unrenormalized meson vertex operator, $\Gamma_{5j}'(p, p, \alpha')$, is defined by

$$\Gamma_{5j}'(p, p, \alpha') \equiv \frac{1}{i} \frac{\partial}{\partial \alpha_j} S'^{-1}(p, \alpha') = \frac{\partial \alpha_k'}{\partial \alpha_j} \{ \tau_k \gamma_5 d - 2i\alpha_k' \\ \times [i\gamma \cdot p \dot{g} + M \dot{f} + \dot{M} f + i\alpha' \cdot \tau \gamma_5 \dot{d}] \}, \quad (2.29)$$

where $\dot{g} = \partial g / \partial \alpha'^2$, etc. Using Eqs. (2.24) one finds immediately,

$$\bar{\psi}'(p, \alpha') \Gamma_{5j}'(p, p, \alpha') \psi'(p, \alpha') = \bar{\psi}'(p, \alpha') \tau_k \gamma_5 \psi'(p, \alpha') \frac{\partial \alpha_k'}{\partial \alpha_j} \\ \times \{ d + 2M \dot{M} f + 2M^2(\dot{f} - \dot{g}) + 2\alpha'^2(\dot{d} - \dot{g}) \}. \quad (2.30)$$

If we substitute for S' its expression in terms of S_c given in Eq. (2.21), we obtain

$$\Gamma_{5j}'(p, p, \alpha') = \frac{\partial \alpha_k'}{\partial \alpha_j} \frac{\Gamma_{5k_c}(p, p, \alpha')}{Z_2(\alpha')} + S_c^{-1}(p, \alpha') - \frac{1}{i} \frac{\partial}{\partial \alpha_j} \frac{1}{Z_2(\alpha')}, \quad (2.31)$$

where we define the convergent vertex operator, $\Gamma_{5k_c}(p, p, \alpha')$, by

$$\Gamma_{5k_c}(p, p, \alpha') = -\frac{1}{i} \frac{\partial}{\partial \alpha_k'} S_c^{-1}(p, \alpha'). \quad (2.32)$$

If we substitute (2.32) into the right-hand side of (2.31), and use Eqs. (2.21) to express the spinor amplitudes in terms of renormalized quantities and use (2.26) for $Z_2(\alpha')$, we find

$$\bar{\psi}_c(p, \alpha') \Gamma_{5k_c}(p, p, \alpha') \psi_c(p, \alpha') = \bar{\psi}_c(p, \alpha') \tau_k \gamma_5 \psi_c(p, \alpha') \times \left\{ \frac{f(1+2M\dot{M}) + 2M^2(\dot{f}-\dot{g}) + 2\alpha'^2(\dot{d}-\dot{g})}{f+2M^2(f'-g') + 2\alpha'^2(d'-g')} \right\}, \quad (2.33)$$

where all quantities are evaluated at $p^2 = -M^2 - \alpha'^2$ and we have used the fact that $d=f$ under this condition. Derivatives of the various functions $f(p^2, \alpha'^2)$, etc., with respect to p^2 and α'^2 , evaluated at $p^2 = -M^2 - \alpha'^2$ can be related to each other using Eq. (2.14) which we rewrite very explicitly as

$$f[-M^2(\alpha'^2) - \alpha'^2, \alpha'^2] = g[-M^2(\alpha'^2) - \alpha'^2, \alpha'^2], \quad (2.34)$$

$$d[-M^2(\alpha'^2) - \alpha'^2, \alpha'^2] = g[-M^2(\alpha'^2) - \alpha'^2, \alpha'^2].$$

Differentiating with respect to α'^2 , we find

$$(f'-g')(1+2M\dot{M}) = \dot{f}-\dot{g}, \quad (2.35)$$

$$(d'-g')(1+2M\dot{M}) = \dot{d}-\dot{g}.$$

Thus Eq. (2.33) reduces to

$$\bar{\psi}_c(p, \alpha') \Gamma_{5k_c}(p, p, \alpha') \psi_c(p, \alpha') = (1+2M\dot{M}) \bar{\psi}_c(p, \alpha') \gamma_5 \tau_k \psi_c(p, \alpha'). \quad (2.36)$$

The presence of the factor $\partial \alpha_k' / \partial \alpha_j$ in Eq. (2.31) and of $(1+2M\dot{M})$ in Eq. (2.36) shows that even in the limit as $\alpha \rightarrow 0$, there is no analog of the Ward identity in meson theory. In terms of the finite functions F , G , and D defined in Eq. (2.27) there are two identities worth recording:

$$F+2M^2(F'-G')+2\alpha'^2(D'-G')=1, \quad (2.37)$$

$$F(1+2M\dot{M})+2M^2(\dot{F}-\dot{G})+2\alpha'^2(\dot{D}-\dot{G})=1+2M\dot{M}. \quad (2.38)$$

These relations play a fundamental role in our later work.

Our renormalization procedure gives, in the limit of $\alpha \rightarrow 0$, precisely the same convergent propagation function $S_c(p)$ as would be computed by the conventional Dyson-Ward procedures. More precisely, our method can be shown to be exactly equivalent to casting S'^{-1} as given by (2.7) into the form

$$S'^{-1}(p, \alpha) = S^{-1}(p, \alpha') + A S^{-1}(p, \alpha') + S^{-1}(p, \alpha') B S^{-1}(p, \alpha'), \quad (2.39)$$

where $S^{-1}(p, \alpha') = i\gamma \cdot p + M + i\alpha' \cdot \tau \gamma_5$, A is a divergent function of M^2 and α'^2 and B is an operator involving $i\gamma \cdot p$, M , and $i\alpha' \cdot \tau \gamma_5$ which is nonsingular on the one-particle energy shell. A is related to $Z_2(\alpha')$ by the relation $Z_2^{-1}(\alpha') = 1 + A$. It is very important to notice that the convergent part of the vertex operator Γ_5 , computed in our scheme, in general, differs from that computed by the usual procedure. As has been pointed out by Kroll and Ruderman, a precise statement of the conventional procedure is that

$$\lim_{p' \rightarrow p} \psi_c(p') \Gamma_{5j}(p', p) \psi_c(p) / \psi_c(p') \gamma_5 \tau_j \psi_c(p) = 1, \quad (2.40)$$

whereas our method is that detailed in Eqs. (2.31) and (2.36). The reason for the ambiguity at this point (which has nothing to do with the ambiguities of mesic charge renormalization discussed in Sec. I) is that as $\alpha \rightarrow 0$, the expectation value of γ_5 in a state of given momentum p vanishes; the usual procedure is based on the limit, as p' approaches p of $\bar{\psi}(p') \gamma_5 \psi(p)$, whereas our method is based on considering the limit, as $\alpha' \rightarrow 0$, of $\bar{\psi}(p, \alpha') \gamma_5 \psi(p, \alpha')$. These two limiting processes are entirely independent of each other. It is perhaps worth remarking that our procedure is well defined with a nonzero value of α' , and consequently has the advantage of avoiding the ratio considerations [see Eq. (2.36)] of the usual method. In a sense, our renormalization scheme is not yet completely specified in that we have said nothing about what shall be called the renormalized coupling constant. A little consideration shows that this is a decision which can be reserved until a later time; the essential infinity to be absorbed in the coupling constant is evidently the quantity $\partial \alpha' / \partial \alpha$ (aside from the usual Z_3) and any finite multiple may be chosen at our convenience. We shall return to this point later on in Secs. III and IV.

Rather than go into detail and show how all divergences are removed in perturbation theory, which would essentially require a repetition of previous work,¹⁵ we shall illustrate our method by a second-order perturbation calculation which will also be useful in later applications.

Using the standard Feynman techniques, the mass operator $\Sigma(p, \alpha)$ [which is related to S'^{-1} by the relation $S'^{-1} = S^{-1} - \Sigma(p, \alpha)$] is easily found to be (to order g^2 ,

¹⁵ P. T. Matthews and A. Salam, *Revs. Modern Phys.* **23**, 311 (1951).

and with the neglect of the mass of the virtual meson)

$$\begin{aligned}\Sigma(p, \alpha) &= \lim_{\lambda \rightarrow \infty} i g^2 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{\lambda^2}{q^2 + \lambda^2} \gamma_5 \tau_k \\ &\quad \times [i\gamma \cdot (p - q) + m + i\alpha \cdot \tau \gamma_5]^{-1} \gamma_5 \tau_k \\ &= \lim_{\lambda \rightarrow \infty} \frac{(-g^2 \lambda^2)}{(4\pi)^2} \int_0^1 dx \int_0^x dy \\ &\quad \times \frac{3i\gamma \cdot p(1-y) + i\alpha \cdot \tau \gamma_5 + 3m}{\Lambda^2}, \quad (2.41)\end{aligned}$$

where

$$\Lambda^2 = (m^2 + \alpha^2)y^2 + \lambda^2(1-x) + (p^2 + m^2 + \alpha^2)y(1-y). \quad (2.42)$$

Reference to Eq. (2.7) shows that (dropping the explicit passage to the limit of the cutoff λ going to infinity)

$$\begin{aligned}f_0(p^2, \alpha^2) &= 1 + \frac{3g^2}{(4\pi)^2} \lambda^2 \int_0^1 dx \int_0^x dy \frac{1}{\Lambda^2}, \\ g_0(p^2, \alpha^2) &= 1 + \frac{3g^2}{(4\pi)^2} \lambda^2 \int_0^1 dx \int_0^x dy \frac{1-y}{\Lambda^2}, \quad (2.43) \\ d_0(p^2, \alpha^2) &= 1 + \frac{g^2}{(4\pi)^2} \lambda^2 \int_0^1 dx \int_0^x dy \frac{1}{\Lambda^2}.\end{aligned}$$

In the limit of α approaching zero, the various quantities that appear in our formalism can be calculated directly from (2.43). We summarize below some of the relevant ones:

$$\begin{aligned}M &= m \left(1 + \frac{3g^2 \lambda^2}{(4\pi)^2} \int_0^1 dx \int_0^x dy \frac{y}{\Lambda_0^2} \right), \\ \frac{\partial \alpha'_k}{\partial \alpha_j} &= \delta_{jk} \left(1 + \frac{g^2 \lambda^2}{(4\pi)^2} \int_0^1 dx \int_0^x dy \frac{3y-2}{\Lambda_0^2} \right), \\ Z_2(0) &= 1 - \frac{3g^2 \lambda^2}{(4\pi)^2} \int_0^1 dx \int_0^x dy \frac{1-y}{\Lambda_0^2} + \frac{3g^2}{(4\pi)^2}, \\ 1 + 2M\dot{M} &= 1 - 3g^2/(4\pi)^2, \\ F(p^2) &= 1 + 3g^2/(4\pi)^2 \\ &\quad + \frac{3g^2 \lambda^2}{(4\pi)^2} \int_0^1 dx \int_0^x dy \left(\frac{1}{\Lambda^2} - \frac{1}{\Lambda_0^2} \right), \\ G(p^2) &= 1 + 3g^2/(4\pi)^2 \\ &\quad + \frac{3g^2 \lambda^2}{(4\pi)^2} \int_0^1 dx \int_0^x dy \left(\frac{1}{\Lambda^2} - \frac{1}{\Lambda_0^2} \right) (1-y), \\ D(p^2) &= 1 + 3g^2/(4\pi)^2 \\ &\quad + \frac{g^2 \lambda^2}{(4\pi)^2} \int_0^1 dx \int_0^x dy \left(\frac{1}{\Lambda^2} - \frac{1}{\Lambda_0^2} \right),\end{aligned} \quad (2.44)$$

where $\Lambda_0^2 = M^2 y^2 + \lambda^2(1-x)$ and Λ^2 is given by (2.42) with α set equal to zero and m replaced by M .

This simple procedure is to be compared with the usual¹⁶ one in which one is required to cast the mass operator into the special form related to (2.39):

$$\begin{aligned}\Sigma(p, \alpha) &= -\frac{g^2 \lambda^2}{(4\pi)^2} \int_0^1 dx \int_0^x dy \left\{ \frac{3my}{\Lambda_0^2} + i\alpha \cdot \tau \gamma_5 \frac{3y-2}{\Lambda_0^2} \right. \\ &\quad + \frac{S^{-1}}{\Lambda_0^2} \left[3(1-y) - \frac{3m^2 y^2 (1-y)}{\Lambda_0^2} - \frac{(3y-2)y(1-y)2\alpha^2}{\Lambda_0^2} \right] \\ &\quad + S^{-1} \frac{y(1-y)}{\Lambda_0^2} \left[\frac{3(1-y)}{\Lambda_2} (i\gamma \cdot p + i\alpha \cdot \tau \gamma_5 - m) \right. \\ &\quad \left. - \frac{3my + (3y-2)i\alpha \cdot \tau \gamma_5}{\Lambda^2} + \frac{6my}{\Lambda_0^2} \right] S^{-1} + \frac{S^{-1} y(1-y)^2}{\Lambda^2 \Lambda_0^4} \\ &\quad \left. \times [6my^2 (i\gamma \cdot p + i\alpha \cdot \tau \gamma_5) - (3y-2)(2\alpha^2)] \right. \\ &\quad \left. \times (i\gamma \cdot p + i\alpha \cdot \tau \gamma_5 - m) S^{-1} \right\}.\end{aligned}$$

III. APPLICATION TO MESON-NUCLEON SCATTERING

We shall now apply the techniques developed in Sec. II to the problem of meson-nucleon scattering. The first application will be a derivation of the exact scattering amplitude in the limit of zero kinetic energy and zero external meson mass. The result obtained will be compared in detail with a perturbation theoretic treatment of the same problem and the precise way in which radiative corrections disappear in our scheme will be exhibited.

The exact scattering amplitude, expressed entirely in terms of unrenormalized quantities, is given by¹⁷

$$\begin{aligned}T_{ji} &= \phi_j^{(-)} \phi_i^{(+)} g_u^2 \lim_{\alpha \rightarrow 0} \bar{\Psi}'(p, \alpha') S'^{-1}(p, \alpha') \\ &\quad \times \left[\frac{1}{i^2} \frac{\partial^2}{\partial \alpha_j \partial \alpha_i} S'(p, \alpha') \right] S'^{-1}(p, \alpha') \Psi'(p, \alpha'). \quad (3.1)\end{aligned}$$

This is the scattering amplitude for the process: meson in charge state i going into meson in charge state j with the appropriate changes in the charge states of the nucleon of momentum p , in the limit of zero total energy mesons; g_u is the unrenormalized coupling constant. Carrying out the indicated differentiations and using the definition (2.29) we obtain

$$\begin{aligned}T_{ji} &= \phi_j^{(-)} \phi_i^{(+)} g_u^2 \lim_{\alpha \rightarrow 0} \bar{\Psi}'(p, \alpha') \left\{ \Gamma_{5j}'(p, p, \alpha') S'(p, \alpha') \right. \\ &\quad \times \Gamma_{5i}'(p, p, \alpha') + \Gamma_{5i}'(p, p, \alpha') S'(p, \alpha') \Gamma_{5j}'(p, p, \alpha') \\ &\quad \left. - \frac{1}{i} \frac{\partial}{\partial \alpha_j} \Gamma_{5i}'(p, p, \alpha') \right\} \Psi'(p, \alpha'). \quad (3.2)\end{aligned}$$

¹⁶ See, for example, R. Karplus and N. M. Kroll, Phys. Rev. **77**, 536 (1950).

¹⁷ This is a special case of the general expression for the scattering amplitude given in reference 7 in terms of variational derivatives.

We now substitute for the unrenormalized quantities their expressions in terms of renormalized quantities. The only renormalization which has not yet been defined is that relating to the meson field.¹⁸ This is, following the usual convention,

$$\phi' = (\sqrt{Z_3})\phi.$$

In order to avoid any possible ambiguity in the evaluation of (3.2), it is convenient to imagine that the momentum p of the internal nucleon lines is kept slightly different from the actual momentum p appearing in the spinors until α has been set equal to zero.

In carrying out the operations indicated in Eq. (3.2), the following facts should be borne in mind: (1) α' , the renormalized external field, must be an odd function of α since it is a pseudoscalar; consequently in the limit as $\alpha \rightarrow 0$, $\partial\alpha'_i/\partial\alpha_j \sim \delta_{ij}$ and we shall write the coefficient of δ_{ij} as simply $\partial\alpha'/\partial\alpha$. (2) The renormalization constant $Z_2(\alpha')$ is a function of α'^2 alone and therefore $\partial Z_2/\partial\alpha'$ is zero in the limit as $\alpha \rightarrow 0$. (3) As $\alpha \rightarrow 0$, the expectation value of γ_5 approaches zero. Using Eqs. (2.29) and (2.31) to evaluate Eq. (3.2) we obtain

$$T_{ji} = \phi_j^{(-)} \phi_i^{(+)} Z_3 g_u^2 \left(\frac{\partial\alpha'}{\partial\alpha} \right)_{\alpha=0}^2 \bar{\psi}_c(p) \times \{ 2\delta_{ij} F^2 \gamma_5 S_c(p) \gamma_5 + 2\delta_{ij} (M\dot{F} + F\dot{M} + M\dot{G}) \} \psi_c(p). \quad (3.3)$$

[We have set $D(-M^2 - \alpha'^2) = F(-M^2 - \alpha'^2)$ since we may now go on to the energy shell.] Recalling that on the energy shell with $\alpha = 0$, $S_c(p) = F^{-1}(i\gamma \cdot p + M)^{-1}$ and that $\bar{\psi}_c \gamma_5 (i\gamma \cdot p + M)^{-1} \gamma_5 \psi_c = (2M)^{-1} \bar{\psi}_c \psi_c$, we have

$$T_{ji} = \phi_j^{(-)} \phi_i^{(+)} Z_3 g_u^2 \left(\frac{\partial\alpha'}{\partial\alpha} \right)_{\alpha=0}^2 \frac{\delta_{ij}}{M} \bar{\psi}_c(p) \times \{ F(1 + 2M\dot{M}) + 2M^2(\dot{F} - \dot{G}) \} \psi_c(p). \quad (3.4)$$

By using (2.38), this result may be written as

$$T_{ji} = \phi_j^{(-)} \phi_i^{(+)} (g_s \delta_{ij} / M) \bar{\psi}_c(p) \psi_c(p), \quad (3.5)$$

where we have defined the renormalized coupling constant g_s^2 to be

$$g_s^2 = g_u^2 Z_3 \left(\frac{\partial\alpha'}{\partial\alpha} \right)_{\alpha=0}^2 (1 + 2M\dot{M}). \quad (3.6)$$

The scattering amplitude as given by Eq. (3.5) is identical with the result of second-order perturbation theory in the limit of zero meson mass with, of course, the replacement of the unrenormalized coupling constant and nucleon mass by the renormalized quantities. The significance of this result will be discussed in Sec. V.

In order to illustrate our formal procedure in detail we present in Table I a summary of the calculation of the same process in second-order perturbation theory.

¹⁸ One may speculate that there exists a close connection between $\sqrt{Z_3}$ and $(\partial\alpha'/\partial\alpha)$ since both are meson field renormalizations.

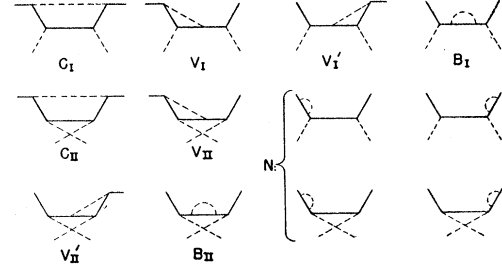


FIG. 1. Feynman diagrams for meson-nucleon scattering. Nucleons are represented by solid lines, mesons by dashed lines.

The appropriate Feynman diagrams are shown in Fig. 1. The calculation is completely standard and one need only use the values of F , G , D , M , and α' given in Sec. II to carry out the evaluation. The results of the conventional procedure of calculating the scattering are also given in Table I.

In the appendix we derive an expression for the μ/M corrections to the scattering amplitude in terms of quantities which can be computed (in principle) from the theory.

IV. THE THREE-FIELD PROBLEM

Having established a method to handle the meson-nucleon problem, and settled upon a renormalization scheme, we now turn to the more involved three-field situation.

We begin by suitably generalizing the technique developed in Sec. II in order to include interactions with the electromagnetic field. The construction of the electric current operator, $j_\mu = ie\bar{\psi}\Gamma_\mu\psi$, is complicated by the fact that charge is exchanged between the meson field and the nucleon. As will be seen in more detail below, the difficulty is not due to the virtual meson field but rather to the charge bearing external field. The direct use of the vertex operator, $\Gamma_\mu(p, p)$, as defined in Eq. (2.18) does not provide us with the desired quantity. The appropriate electric current vertex operator Γ_μ can be constructed by variational differentiation of the pro-

TABLE I. Lowest order corrections to meson-nucleon scattering.^a

Method	Contribution	Type of graph			
		$C_I + C_{II}$	$V_I + V_{II}$	$V'_I + V_{II}'$	$B_I + B_{II} + N$
α formalism	Finite	-6	0	0	3
	Infinite	0	1	1	$6y-6$
Conventional	Finite renorm.	0	7/6	7/6	6
	Infinite renorm.	0	1	1	$6y-6$
	Finite correction	-6	2	2	-3

^a All finite terms are multiplied by $g^2/(4\pi)^2$, all infinite ones by $[g^2/(4\pi)^2] \cdot \int_0^1 dx \int_0^1 dy \lambda^2 / [m^2 y^2 + \lambda^2(1-x)]$, both times the lowest order matrix element. Types of graph refer to Fig. 1. It will be noted that the sum of all terms in the α formalism is precisely $(\partial\alpha'/\partial\alpha)^2(1+2M\dot{M})$ to this order, as can be seen from Eq. (2.44). The infinite contributions, but not the finite ones, are the same in each column for both methods; the conventional method yields finite corrections to scattering. Effects from vacuum polarization, giving rise to an over-all Z_1 factor, have been omitted.

propagation function with respect to an external vector potential $A_\mu(x)$ together with consistent use of the isotopic spin formalism. As we shall be interested only in low energy phenomena we restrict our attention to a constant external potential A_μ , so that variational derivatives may be replaced by ordinary derivatives. A convenient way of introducing such a potential into the propagation function is by a gauge transformation:

$$S'^{-1}(p, \alpha') \rightarrow S'^{-1}(p, \alpha', A_\mu) = \exp(iS) S'^{-1}(p, \bar{\alpha}') \exp(-iS), \quad (4.1)$$

where

$$S = \frac{1}{2}(1 + \tau_3) x_\mu A_\mu, \quad (4.2)$$

$$\bar{\alpha}' \cdot \tau = \exp(-iS) \alpha' \cdot \tau \exp(iS),$$

and x_μ is to be interpreted as a differential operator, $i\partial/\partial p_\mu$. In the absence of the external meson potential α , the sole effect of the transformation S would have been to replace p_μ by the gauge invariant combination $p_\mu - \frac{1}{2}(1 + \tau_3)A_\mu$; this must still be true in the more general case, hence the replacement of α' by $\bar{\alpha}'$. Since the transformation is a rotation in isotopic spin space, α'^2 , a scalar, is of course equal to $\bar{\alpha}'^2$.

At this point we must recall that, of the terms permitted by invariance and charge conjugation considerations in $S'^{-1}(p, \alpha')$, one, having the form $\{i\gamma \cdot p, i\alpha' \cdot \tau \gamma_5\}h$ was identically zero. Such is no longer the case when α' is replaced by $\bar{\alpha}'$, since $\bar{\alpha}'$ and p no longer commute, and this term must be reinstated. Further, the term $i\bar{\alpha}' \cdot \tau \gamma_5 d$ must now be properly symmetrized because $\bar{\alpha}'$ and d do not commute. Therefore, according to Eq. (4.1) and the above remarks, our complete propagation function becomes

$$S_c^{-1}(p, \alpha', A) = \exp(iS) \left[i\gamma \cdot p G(p^2, \alpha'^2) + M F(p^2, \alpha'^2) + \frac{i\gamma_5}{2} \{ \bar{\alpha}' \cdot \tau, D(p^2, \alpha'^2) \} + \frac{1}{4} \left[H, \left\{ \frac{i\gamma \cdot p}{M}, i\bar{\alpha}' \cdot \tau \gamma_5 \right\} \right] \right] \exp(-iS). \quad (4.3)$$

(None of the renormalization considerations of Sec. II are affected since $\alpha'^2 = \bar{\alpha}'^2$, so we discuss only the convergent function.) We now define the electric current vertex operator $\Gamma_{\mu c}(p, p, \alpha', A)$ as follows:

$$\begin{aligned} \Gamma_{\mu c}(p, p, \alpha', A) &= i \frac{\partial S_c^{-1}}{\partial A_\mu} = \exp(iS) \left[\frac{1 + \tau_3}{2} \gamma_\mu G - \frac{1 + \tau_3}{2} 2i p_\mu \right. \\ &\quad \times (i\gamma \cdot p G' + M F') + \gamma_5 p_\mu D'(\bar{\alpha}' + \alpha' \cdot \tau) \\ &\quad + \frac{iH}{4M} \gamma_\mu \gamma_5 [\tau_3, \bar{\alpha}' \cdot \tau] + \frac{1 + \tau_3}{2} 2i p_\mu \frac{H'}{4M} \\ &\quad \left. \times \{i\gamma \cdot p, i\bar{\alpha}' \cdot \tau \gamma_5\} \right] \exp(-iS). \quad (4.4) \end{aligned}$$

In the limit as α' and A_μ approach zero, $\Gamma_{\mu c}(p, p)$ is indeed given by $\frac{1}{2}(1 + \tau_3)(\partial/i\partial p_\mu) S_c^{-1}(p)$ and is thus [except for the factor $\frac{1}{2}(1 + \tau_3)$] the same as the $\Gamma_\mu(p, p)$ of Sec. II. The actual vertex operator in the presence of the α field is, of course, the limit of Eq. (4.4) as $A_\mu \rightarrow 0$; the complications introduced by the external field are evident.

The meson vertex operator $\Gamma_{5jc}(p, p, \alpha', A)$ is defined as before as the derivative of S_c^{-1} with respect to $i\alpha'_j$. In the limit of $A_\mu \rightarrow 0$, it approaches (2.32).

We may remark at this point that the transition matrix element for the zero frequency Compton effect on a nucleon may be written as

$$T_{\lambda\lambda_0} = e_0^2 A_\mu'^{(-)}(\lambda) \bar{\psi}'(p) \lim_{A \rightarrow 0} \left[S'^{-1}(p) \left\{ i^2 \frac{\partial^2}{\partial A_\mu \partial A_\nu} S'(p) \right\} \times S'^{-1}(p) \right] \psi'(p) A_\nu'^{(+)}(\lambda_0), \quad (4.5)$$

where λ, λ_0 are polarization indices, e_0 is the unrenormalized electric charge, $A'^{(\pm)}$ are the unrenormalized amplitudes of the photons. The external meson field may be set equal to zero, consequently there are no isotopic spin complications. An over-all factor of $\frac{1}{2}(1 + \tau_3)$ insures that in the limit of zero frequency, the light scattering from a neutron vanishes. For a proton, $\frac{1}{2}(1 + \tau_3) = +1$ and the derivatives with respect to the external field A may be replaced by derivatives with respect to the momentum p , i.e., $\partial/\partial(-iA_\mu) = \partial/\partial i p_\mu$. To evaluate Eq. (4.7) we replace the unrenormalized quantities by renormalized ones, noting that all $Z_2(0)$'s cancel out and that as far as the photon amplitudes are concerned, we must take $e_0 A' = e_1 A$, where e_1 and A are the renormalized quantities. Using (4.4), with α' and A equal to zero, and choosing a transverse gauge to describe the photons, so that all terms proportional to p_μ vanish, we find for $T_{\lambda\lambda_0}$ the result:

$$\begin{aligned} T_{\lambda\lambda_0} &= \frac{e_1^2}{M} A_\mu'^{(-)}(\lambda) A_\mu'^{(+)}(\lambda_0) \\ &\quad \times [F + 2M^2(F' - G')]_{p^2 = -M^2} \bar{\psi}_c(p) \psi_c(p) \\ &= \frac{e_1^2}{M} A_\mu'^{(-)}(\lambda) A_\mu'^{(+)}(\lambda_0) \bar{\psi}_c(p) \psi_c(p). \quad (4.6) \end{aligned}$$

The last step follows from Eq. (2.37). This is not a new result and has been included simply to show precisely how it comes out in our formalism, which differs in detail from the earlier work of Thirring and of Kroll and Ruderman.

We turn now to the problem of threshold photomeson production in the limit of vanishing meson mass, the problem discussed by Kroll and Ruderman. There is

one slight complication in this process which must be considered before we can perform the explicit calculation. We must allow for the interaction between the electromagnetic field (photon) and the meson produced. We shall not go into this question exhaustively because, as will be seen, the meson current contributions vanish at threshold. If the propagation function for a meson of four-momentum q , $\Delta_{ij}'(q)$, (i and j are isotopic spin indices) is known, the effect of a constant vector potential may be introduced conveniently by a gauge transformation:

$$\Delta_{ij}'(q, A) = (e^{iT \cdot A} \Delta'(q) e^{-iT \cdot A})_{ij}, \quad (4.7)$$

where T is the matrix

$$T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (4.8)$$

and x_μ is to be interpreted as $i\partial/\partial q_\mu$. The transition matrix $T_{j\lambda}$ for the threshold photo production of a meson of isotopic spin j by a photon of polarization λ in the limit of vanishing meson mass, is given by

$$T_{j\lambda} = e_0 g_u \lim_{\alpha, q_\mu, A_\mu \rightarrow 0} \bar{\psi}'(p, \alpha') \phi_k'^{(-)}(j) S'^{-1}(p, \alpha', A) \times \Delta_{kl}^{-1}(q, A) \left\{ \frac{\partial}{\partial A_\mu} \left[\frac{\partial S'(p, \alpha', A)}{\partial \alpha_n} \Delta_{ln}'(q, A) \right] \right\} \times S'^{-1}(p, \alpha', A) A_\mu'^{(+)}(\lambda) \psi'(p, \alpha'). \quad (4.9)$$

All repeated indices in Eq. (4.9) are to be summed over; in the subsequent equations our notation will be less specific, e.g., $\phi_k(j)$ will be set simply equal to ϕ_j as was done in Sec. III. We now remark that the term arising from the meson current, i.e., from the derivative of Δ_{ln}' , vanishes in our limit since, by invariance, it can only be proportional to q_μ .¹⁹ In terms of our previously defined vertex operators, we may write this as

$$T_{j\lambda} = e_0 g_u \lim_{\alpha, q_\mu, A_\mu \rightarrow 0} \bar{\psi}'(p, \alpha') \phi_j'^{(-)} [\Gamma_{5j}' S' \Gamma_\mu' + \Gamma_\mu' S' \Gamma_{5j}'] - \partial^2 S'^{-1} / \partial \alpha_j \partial A_\mu \bar{\psi}'(p, \alpha') A_\mu'^{(+)}(\lambda). \quad (4.10)$$

The arguments of the vertex operators are all p, p, α', A and of the propagation function p, α', A [see Eqs. (4.3) and (4.4)]. The renormalized functions may now be introduced in the usual way. The third term in (4.10) makes the contribution

$$\frac{\partial^2 S_c^{-1}}{\partial \alpha_j \partial A_\mu} = \left(\frac{\partial \alpha'}{\partial \alpha} \right) \frac{1}{2M} \gamma_\mu \gamma_5 (\tau T)_j H, \quad (4.11)$$

where $(\tau T)_j = \tau_i T_{ij}$. The first two terms yield (aside from a factor of $\partial \alpha' / \partial \alpha$), in the limit as α, A approach

¹⁹ M. Kroll and M. A. Ruderman have shown more generally that the outgoing meson current does not contribute to threshold production.

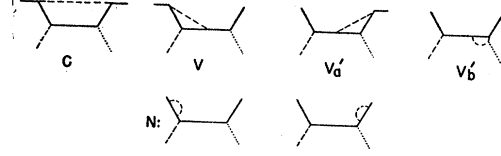


FIG. 2. Feynman diagrams for photomeson production. Nucleons are represented by solid lines, mesons by dashed lines, and photons by dotted lines.

zero, $(i/2)(\gamma_\mu \gamma_5 / M)(\tau T)_j F$. We obtain, finally,

$$T_{j\lambda} = e_1 g_u \left(\frac{\partial \alpha'}{\partial \alpha} \right)_{\alpha=0} Z_3^{1/2} (F+H) \phi_j^{(-)} A_\mu^{(+)}(\lambda) \bar{\psi}_c(p) \frac{i}{2M} \times \gamma_\mu \gamma_5 (\tau T)_j \psi_c(p).$$

If we were to choose our renormalized coupling constant to be

$$g_P = (\partial \alpha' / \partial \alpha) Z_3^{1/2} (F+H) g_u, \quad (4.13)$$

we would obtain no finite correction to the photoproduction amplitude. The numerical value of g_P , obtained by comparison with experiment is of course the same as that found by Kroll and Ruderman. It should be carefully borne in mind, however, that the precise structure of g_P , regarded as a theoretically calculable quantity, is dependent on the procedure followed in carrying out the meson vertex renormalizations. The form given in Eq. (4.13) is what results from using our computational schemes throughout. This point has been discussed at the end of Sec. II and is illustrated in detail in Table II where an explicit comparison between the two procedures is presented. The appropriate Feynman diagrams are shown in Fig. 2.

The relation between the coupling constants g_s [defined in Eq. (3.6)] and g_P is seen to be

$$g_s^2 / g_P^2 = (1 + 2M\dot{M})(F+H)^{-2}. \quad (4.14)$$

Computing these functions in lowest order perturbation theory, we get for this ratio:

$$g_s^2 / g_P^2 = 1 - 5g_s^2 / 16\pi^2. \quad (4.15)$$

TABLE II. Lowest-order corrections to photomeson production.^a

Method	Contribution	C	V	Type of graph $V_a' + V_b'$	B	N
α formalism	Finite	-2	0	0	0	3
	Infinite	0	1	-3y+3	3y-3	3y-3
Conventional	Finite renorm.	0	7/6	-3	3	3
	Infinite renorm.	0	1	-3y+3	3y-3	3y-3
	Finite correction	-2	2	1+2	-3	0

^a All finite terms are multiplied by $g^2/(4\pi)^2$, all infinite ones by $[g^2/(4\pi)^2] \cdot \int_0^1 dx \int_0^1 dy \lambda^2 / [m^2 y^2 + (1-x)\lambda^2]$, both times the lowest order matrix element. Types of graph refer to Fig. 2. It can be seen that there are no net corrections in the conventional method. The infinite contributions are the same in both cases, but there is a finite correction in the α formalism, as the first two lines do not add up to $(\partial \alpha' / \partial \alpha)(1 + 2M\dot{M})^{1/2}$ but rather to $(\partial \alpha' / \partial \alpha)(F+H)$. Vacuum polarization effects have been omitted.

The above result is consistent with the calculations of scattering and photoproduction as given in Tables I and II. If one uses the conventional meson vertex renormalization and the photoproduction definition of mesic charge, one finds that the meson scattering matrix element is proportional to $g_P^2(1-5g_P^2/(4\pi)^2)$, which (to this order) is just equal to g_s^2 . Similarly, if one calculates photomeson production with our vertex renormalization and uses the meson scattering definition of mesonic charge, one obtains a matrix element proportional to $g_s[1+5/2g_s^2(4\pi)^2]$, which is just equal to g_P by Eq. (4.15).

V. COMPARISON WITH EXPERIMENT AND DISCUSSION

We may now compare the various results obtained in the previous sections with experiment. First, we shall determine the value of our renormalized coupling constant, and second, we shall investigate, in so far as it is possible, whether or not the resultant theory is in agreement with experimental data.

The general expression for the scattering matrix describing the forward scattering of a meson of four-momentum q by a nucleon of four-momentum p is⁹

$$T_{ji} = \frac{g_s^2}{2M} \left\{ 2\delta_{ji} A(q^2/M^2, \mu^2/M^2) - \frac{p \cdot q}{2M^2} [\tau_i, \tau_j] B(q^2/M^2, \mu^2/M^2) \right\}, \quad (5.1)$$

where we do not write the various field amplitudes explicitly. We have written the functions A and B with two arguments to emphasize that the external and internal meson masses need not be assumed equal. Designating temporarily the external meson mass by μ' , we obtain, for zero energy, the scattering lengths corresponding to isotopic spin states $3/2$ and $1/2$:

$$\begin{aligned} a_{3/2} &= -\frac{g_s^2}{4\pi M} \left\{ A\left(\frac{\mu'^2}{M^2}, \frac{\mu^2}{M^2}\right) + \frac{\mu'}{2M} B\left(\frac{\mu'^2}{M^2}, \frac{\mu^2}{M^2}\right) \right\}, \\ a_{1/2} &= -\frac{g_s^2}{4\pi M} \left\{ A\left(\frac{\mu'^2}{M^2}, \frac{\mu^2}{M^2}\right) - \frac{\mu'}{2M} B\left(\frac{\mu'^2}{M^2}, \frac{\mu^2}{M^2}\right) \right\}. \end{aligned} \quad (5.2)$$

In our renormalization scheme, $A(0, \mu^2/M^2) = 1$; if we assume that $A(\mu'^2/M^2, \mu^2/M^2)$ may be approximated by $A(0, \mu^2/M^2)$, we may solve Eqs. (5.2) for $g_s^2/4\pi$ and B . Little can be said as to the reliability of such an approximation. In lowest order perturbation theory A is $[1 - \mu'^2/4M^2]^{-1}$, so that the approximation is very good. As far as higher terms are concerned, the numerical value of the corrections depends on the size of the coupling constant. In the appendix, a formal expression for $B(0, \mu^2/M^2)$ will be derived.

Before going on to the experimental determination of $g_s^2/4\pi$ by means of Eqs. (5.2), attention must be drawn to one point concerning the definition of g_s^2 . It

will be recalled that there appeared in this definition (aside from inherently positive factors) the quantity $(1+2MM)$. Its behavior as a function of M and g_s is, of course, unknown in practice. This is in contradistinction to electrodynamics where the theory is known to predict a definite sign for the Compton scattering amplitude. If it were possible to obtain the sign of $(1+2MM)$, then a severe test of the theory could be made with just the scattering data, as the signs of the scattering lengths are known. That is, a direct contradiction could exist with experiment if the predicted sign turned out to be wrong. We have not been able to determine the sign of $(1+2MM)$, however, and will therefore take it as dictated by experiment.

Unfortunately there do not exist measurements of meson-nucleon scattering cross sections at sufficiently low energies to permit an unambiguous extrapolation to zero energy. Fermi and Steinberger have each given extrapolated scattering lengths based on analyses of the data for scattering at energies greater than about 40 Mev. These sets have both been adjusted to agree with the π^- capture data.²⁰ This effect relates charge-exchange scattering and photoproduction at zero energy and leads to the requirement that

$$|a_{3/2} - a_{1/2}| = 0.17/\mu. \quad (5.3)$$

- (a) Fermi²¹ gives $a_{3/2} = -0.11/\mu$, $a_{1/2} = 0.06/\mu$. This implies that $g_s^2/4\pi = 0.36$, taking $1+2MM > 0$.
- (b) Steinberger²² gives $a_{3/2} = 0$, $a_{1/2} = +0.17/\mu$. This leads to $g_s^2/4\pi = +0.37$. Here, however, we had to take $1+2MM < 0$.

Recent preliminary observations at Columbia on π^- scattering at 7 Mev, however, imply that $g_s^2/4\pi$ may be perhaps as large as 1 or 2.²³

It seems striking that all the above results yield far smaller $g_s^2/4\pi$ than usually employed for the pseudoscalar coupling constant, and in particular far smaller than the value of 25 one obtains for $g_P^2/4\pi$ by adjusting it to agree with the photomeson production data.²⁴ We obtain very small values for $g^2/4\pi$ since our renormalization scheme is based on matching the s -wave scattering amplitudes which are experimentally of the order of $1/M$, just the size of the Born approximation amplitudes if $g^2/4\pi$ is about unity.

It is frequently claimed²⁵ that the large values of the

²⁰ Panofsky, Aamodt, Hadley, and Phillips, Phys. Rev. **80**, 94 (1950).

²¹ E. Fermi (private communication).

²² J. Steinberger (private communication).

²³ Corneliuss, Sargent, Rinehart, Lederman, and Rogers, Phys. Rev. **92**, 1583 (1953).

²⁴ It may be mentioned that with a pseudovector coupling theory (assuming that the theory exists in some sense) the renormalization to the meson scattering experiment would lead to a pseudovector coupling constant f_s equal (since this is the lowest order perturbation theory requirement) to our g_s . This is to be contrasted with the photomeson situation where one has the equivalence relation $f_P = g_P(\mu/2M)$. Thus, whereas we had previously had $g_P > g_s$, in this new situation one would have $f_s \geq f_P$.

²⁵ S. Drell and E. Henley, Phys. Rev. **88**, 1053 (1952); G. Wentzel, Phys. Rev. **86**, 802 (1953).

coupling constant which are presumably needed to explain the p wave meson scattering can still yield the correct s scattering at low energies, provided one computes more accurately than Born approximation. This belief is based on the observation that the scattering length obtained from an interaction of the form $g^2\phi^2/2M$ (which is supposed to represent, for low energies, the s wave interaction in pseudoscalar theory) is, for $g^2/4\pi \gg 1$, given by the cut-off radius independent of $g^2/4\pi$. It should be pointed out that the analogous procedure in electrodynamics, in the problem of low-frequency Compton scattering (where the interaction is described by $e^2A^2/2M$) would again lead for large e^2 to a scattering length which does not exceed the cut-off radius. However, as we have seen, the exact value predicted by the theory is e^2/M , regardless of the size of e^2 . This seems to indicate that the use of just the ϕ^2 part of the equivalent pseudoscalar interaction may well be incorrect.

The expectation that one would find small s scattering in an accurate treatment of pseudoscalar theory due to suppression of pair effects²⁶ is belied by the observation that the damping introduced through the propagation function is (for low energies) precisely cancelled by the vertex operator. It is easy to show, using the results of Sec. II that the following equations²⁷ are true:

$$\begin{aligned} S_c(p)\Gamma_{5j_c}(p,p)\psi_c(p) &= S(p)\tau_j\gamma_5\psi_c(p), \\ \bar{\psi}_c(p)\Gamma_{5j_c}(p,p)S_c(p) &= \bar{\psi}_c(p)\tau_j\gamma_5S(p). \end{aligned} \quad (5.4)$$

These results do not depend upon our choice of the coupling constant.

Two alternative renormalization methods exist, as we have seen, corresponding to fixing the coupling constant by two different experiments. If the theory is assumed to be consistent with both experiments, then the method of renormalization yielding the smaller coupling constant may well be of considerable practical use; it effectively re-sums the series in such a way that the expansion in powers of the renormalized coupling constant may have some significance, albeit perhaps only in an asymptotic sense. If, however, one computes in perturbation theory, using our small values of $g_s^2/4\pi$, the threshold photomeson production, one obtains disagreement with experiment by at least an order of magnitude. If one uses the value $g_s^2/4\pi \sim \frac{1}{3}$, the terms in the series are decreasing very rapidly; with $g_s^2/4\pi \sim 2$, the decrease is so slow that one cannot say much.²⁸

²⁶ Brueckner, Gell-Mann, and Goldberger, Phys. Rev. **90**, 476 (1953).

²⁷ Equations (5.4) are analogous to the results of Kroll and Ruderman for $S_c\Gamma_\mu$ and bear out their conjectures as to the behavior of products like $S_c\Gamma_5$.

²⁸ It is clear that a perturbation calculation of meson-nucleon scattering using $g_s^2/4\pi \approx \frac{1}{3}$ fails to reproduce the nonzero energy data. On the other hand, a similar calculation of the π^0 lifetime does agree with the observed value. With $g_s^2/4\pi \approx 2$, the apparent convergence is again too slow to enable one to reach any conclusions.

It is evidently desirable to obtain accurate experimental data on very low energy meson scattering. If the coupling constant as determined by scattering is indeed as small or perhaps even smaller than the value of about $\frac{1}{3}$ given by the Fermi and Steinberger extrapolations, the outlook for pseudoscalar meson theory with pseudoscalar coupling seems dim.

In conclusion, it is to be emphasized that all of our numerical results have been based on the assumption that $A(0, \mu^2/M^2)$ is a good approximation to $A(\mu^2/M^2, \mu^2/M^2)$. Very little can be said at present as to the legitimacy of the assumption. On the other hand, if such a small quantity as $(\mu/M)^2$ is not negligible, it is difficult to see what can be neglected in any given calculation within the theory. Perturbation theory calculations indicate that (with our small coupling constants) the error is small, but this point deserves further study.

ACKNOWLEDGMENTS

We are indebted to Dr. N. M. Kroll and Dr. M. A. Ruderman for sending us a preprint of their paper, and to Dr. Kroll for several enlightening discussions. We have also benefited from conversations with Dr. G. Källén and with Dr. R. Oppenheimer. Thanks are due Drs. E. Fermi and J. Steinberger for discussions of current experiments. Two of us (S.D. and W.T.) are grateful to the Institute for Advanced Study for grants-in-aid, and to Dr. Oppenheimer for the hospitality extended to us at the Institute.

APPENDIX

The evaluation of the μ/M terms, that is of $B(0, \mu^2/M)$ in the notation of Sec. V, breaks naturally into two separate parts, namely those diagrams in which there is at least one free nucleon line (aside from the external meson lines), and those for which there is never a free nucleon line. (The general structure of the scattering amplitude is of this form as can be seen from the exact expression given by Deser and Martin.⁷) Since we are still interested in the scattering amplitude at zero kinetic energy, it is sufficient to calculate the scattering amplitude in the forward direction, the scattering being entirely in s states. Let us take the two classes of terms referred to above separately.

The total contribution of the first class to the forward scattering amplitude (for a meson of initial and final four-momentum q) is given exactly by

$$\begin{aligned} T_{ji}^{(1)} &= \phi_j^{(-)}\phi_i^{(+)}g_u^2 \lim_{\alpha \rightarrow 0} \bar{\psi}'(p, \alpha') \\ &\times \{ \Gamma_{5j'}(p, p+q, \alpha')S'(p+q, \alpha')\Gamma_{5i'}(p+q, p, \alpha') \\ &+ \Gamma_{5i'}(p, p-q, \alpha')S'(p-q, \alpha')\Gamma_{5j'}(p-q, p, \alpha') \} \\ &\times \psi'(p, \alpha'). \end{aligned} \quad (A.1)$$

This expression is correct for any value of q ; we shall, however, in the following keep only the zero and first order terms in q . The zero-order term yields, of course, the first two terms in Eq. (3.2) and essentially requires no further discussion. In order to proceed with the evaluation, we must derive expressions for the $\Gamma_{\bar{s}}$'s of unequal arguments. The S_F 's present no difficulty since we have an explicit expression for them which may be expanded in a straightforward way. It is sufficient to work with the first term in (A.1) since the second follows by interchanging i and j and replacing q by minus q . Since there are no differentiations with respect to α to be performed, we may safely pass to the limit $\alpha \rightarrow 0$ and retain only the obviously non-vanishing terms. Expressing everything in terms of the renormalized quantities we have for the portion under discussion the following:

$$t_{ji} = \phi_j^{(-)} \phi_i^{(+)} g_u^2 \left(\frac{\partial \alpha'}{\partial \alpha} \right)_{\alpha=0}^2 Z_3 \bar{\psi}_c(p) \times \{ \Gamma_{\bar{s}j}(p, p+q) S_c(p+q) \Gamma_{\bar{s}i}(p+q, p) \} \psi_c(p). \quad (\text{A.2})$$

(We have dropped the subscript c from the renormalized vertex operators.) We may write $S_c(p+q)$ explicitly as

$$S_c(p+q) = \frac{MF[(p+q)^2] - i\gamma \cdot (p+q)G[(p+q)^2]}{(p+q)^2 G^2[(p+q)^2] + M^2 F^2[(p+q)^2]}. \quad (\text{A.3})$$

By using the facts that $p^2 = -M^2$ and $q^2 = -\mu^2$, $S_c(p+q)$ may easily be expanded in powers of q . It is necessary to keep only terms which are quadratic in q in both numerator and denominator; although one does not obtain in this way all of the terms linear in q , those which are omitted make no contribution to the matrix element. Designating derivatives of F and G with respect to p^2 , evaluated at $P^2 = -M^2$, by primes, we find for the relevant terms:

$$S_c(p+q) \simeq \frac{1}{F(2p \cdot q - \mu^2)} \left[(-i\gamma \cdot p + M)F - i\gamma \cdot qF \right. \\ + (2p \cdot q - \mu^2)(-i\gamma \cdot pG' + MF') - i\gamma \cdot q(2p \cdot q - \mu^2)G' \\ + \frac{1}{2}(2p \cdot q - \mu^2)^2(-i\gamma \cdot pG'' + MF'') \\ \left. - (2p \cdot q - \mu^2) \left\{ 2G' + M^2 \left(F'' - G'' + \frac{F'^2 - G'^2}{F} \right) \right\} \right] \\ \times \{ (2p \cdot q - \mu^2)(-i\gamma \cdot pG' + MF') - Fi\gamma \cdot q \}. \quad (\text{A.4})$$

The general structure of the off-diagonal elements of $\Gamma_{\bar{s}}$ may be inferred from invariance considerations and certain inter-relations among the coefficients may be deduced from the behavior of $\Gamma_{\bar{s}}$ under charge conjuga-

tion. The charge conjugation requirement is

$$C^{-1} \Gamma_{\bar{s}i}(p_1, p_2) C = -\Gamma_{\bar{s}i}^T(-p_2, -p_1), \quad (\text{A.5})$$

where C is the same matrix defined in Sec. II, and the transposition operation refers both to ordinary spin and isotopic spin indices. One may then show easily that to first order in ϵ_1 , and ϵ_2 ,

$$\Gamma_{\bar{s}i}(p+\epsilon_1, p+\epsilon_2) = \Gamma_{\bar{s}i}(p, p) + (\epsilon_1 + \epsilon_2) \cdot p \tau_i \gamma_5 \frac{H_1(p^2)}{M^2} \\ + (\epsilon_1 - \epsilon_2) \cdot p \tau_i \gamma_5 i\gamma \cdot p \frac{H_2(p^2)}{M^3} + i(\epsilon_1 - \epsilon_2) \cdot \gamma \tau_i \gamma_5 \frac{H_3(p^2)}{M} \\ + \tau_i \gamma_5 (\epsilon_1 - \epsilon_2)_\mu \sigma_{\mu\nu} p_\nu \frac{H_4(p^2)}{M^2}. \quad (\text{A.6})$$

It is evident that the term proportional to $(\epsilon_1 + \epsilon_2)$ may be expressed entirely in terms of $\Gamma_{\bar{s}}$'s of equal arguments, in fact:²⁹

$$(\epsilon_1 + \epsilon_2) \cdot p \tau_i \gamma_5 H_1(p^2)/M = \frac{1}{2} \{ \Gamma_{\bar{s}i}(p+\epsilon_1, p+\epsilon_1) \\ + \Gamma_{\bar{s}i}(p+\epsilon_2, p+\epsilon_2) - 2\Gamma_{\bar{s}i}(p, p) \}. \quad (\text{A.7})$$

We may replace the unknown functions introduced in Eq. (A.6) by previously defined ones by means of certain identities between the off-diagonal elements of $\Gamma_{\bar{s}}$ and the derivatives with respect to A of the diagonal part of $\Gamma_{\bar{s}}$. These identities hold only to first order in q , and are deduced by comparing the method of Sec. IV and that used by Kroll and Ruderman for tracing the charge in the photoproduction calculation. That is, one finds (to first order in q):

$$\Gamma_{\bar{s}-}(p+q, p) = \Gamma_{\bar{s}-}(p, p) - q_\mu \frac{\partial}{\partial A_\mu} \langle P | \Gamma_{\bar{s}-}(p, p) | N \rangle |_{A=0}, \\ \Gamma_{\bar{s}+}(p, p+q) = \Gamma_{\bar{s}+}(p, p) - q_\mu \frac{\partial}{\partial A_\mu} \langle N | \Gamma_{\bar{s}+}(p, p) | P \rangle |_{A=0}, \quad (\text{A.8})$$

where \pm and 0 indices on $\Gamma_{\bar{s}}$ refer to the creation of negative, positive, and neutral mesons, respectively, and the labeling of the matrix elements refers to the charge state of the nucleon ($\tau_3 = \pm 1$ for p and n , respectively). No such relation holds for $\Gamma_{\bar{s}0}$, but rather one involving $\Gamma_{\bar{s}0}$ of equal arguments only. By use of Eq. (A.8) we can deduce by comparison with Eq. (A.6) that

$$H_2 = H_4 = 0, \\ H_1 = M^2 D', \\ H_3 = H. \quad (\text{A.9})$$

The evaluation of $T_{ji}^{(1)}$ can now be carried out in a straightforward fashion. One obtains finally [making frequent use of the identity $F + 2M^2(F' - G') = 1$] for

²⁹ The technique outlined above was developed by M. Gell-Mann and M. L. Goldberger in another connection (unpublished).

$T_{ji}^{(1)}$ the value:

$$T_{ji}^{(1)} = \phi_j^{(-)} \phi_i^{(+)} \frac{g_u^2}{M} Z_3 \left(\frac{\partial \alpha'}{\partial \alpha} \right)_{\alpha=0}^2 \bar{\psi}_c(p) \psi_c(p) \left\{ F \delta_{ji} + \frac{[\tau_j, \tau_i]}{4} \frac{p \cdot q}{M^2} [1 + 4M^2(D' - F') - 4H] \right\}. \quad (\text{A.10})$$

The first term in brackets is, of course, our old result. We turn now to the somewhat more difficult task of evaluating to first order in q those diagrams for which there is never a free nucleon line.

In the notation of Deser and Martin these are the terms described by $\delta\Gamma_5/\delta\phi$. Careful consideration of the various diagrams which contribute to the class under consideration enables one to conclude that, to first order in q , the entire contribution of all these terms is

$$T_{ji}^{(2)} = \phi_j^{(-)} \phi_i^{(+)} g_u^2 \left(\frac{\partial \alpha'}{\partial \alpha} \right)_{\alpha=0}^2 Z_3 \bar{\psi}_c(p) \times \left\{ 2\delta_{ji}(M\dot{F} + \dot{M}F - M\dot{G}) - \frac{1}{i} \frac{\partial}{\partial \alpha_j'} [\Gamma_{5i}(p+q, p, \alpha') - \Gamma_{5i}(p, p, \alpha')] - \frac{1}{i} \frac{\partial}{\partial \alpha_i'} [\Gamma_{5j}(p-q, p, \alpha') - \Gamma_{5j}(p, p, \alpha')] \right\} \psi_c(p). \quad (\text{A.11})$$

We have gone directly to the renormalized quantities; the first term is just our previous one and the last two are the essentially new ones which are correct to first order in q . The identities (A.8) are unfortunately of no use in evaluating these terms, since their only non-vanishing contributions, being proportional to $[\tau_i, \tau_j]$ involve Γ_{50} . We are, therefore, reduced to finding the off-diagonal elements of Γ_5 , for $\alpha \neq 0$, by the procedure outlined above, except that the charge conjugation relation now takes the form

$$C^{-1} \Gamma_{5i}(p_1, p_2, \alpha') C = -\Gamma_{5i}^T(-p_2, -p_1, -\alpha'). \quad (\text{A.12})$$

One finds

$$\begin{aligned} \Gamma_{5j}(p+q, p, \alpha') &= \Gamma_{5j}(p, p, \alpha') + \frac{1}{2} [\Gamma_{5j}(p+q, p+q, \alpha') \\ &\quad - \Gamma_{5j}(p, p, \alpha')] + \frac{i\gamma \cdot q}{M} \tau_j \gamma_5 H_3(p^2, \alpha'^2) + \frac{p \cdot q}{M^2} [i\tau_j, \alpha' \cdot \tau] \\ &\quad \times H_5(p^2, \alpha'^2) + \frac{p \cdot q}{M^2} \frac{i\gamma \cdot p}{M} [i\tau_j, \alpha' \cdot \tau] H_6(p^2, \alpha'^2) \\ &\quad + \frac{i\gamma \cdot q}{M} [i\tau_j, \alpha' \cdot \tau] H_7(p^2, \alpha'^2). \end{aligned} \quad (\text{A.13})$$

If we evaluate (3.16) we obtain

$$\begin{aligned} T_{ji}^{(2)} &= \phi_j^{(-)} \phi_i^{(+)} g_u^2 \left(\frac{\partial \alpha'}{\partial \alpha} \right)_{\alpha=0}^2 Z_3 \bar{\psi}_c(p) \\ &\quad \times \left\{ 2\delta_{ji}(M\dot{F} + \dot{M}F - M\dot{G}) - \frac{2p \cdot q}{M^2} [\tau_i, \tau_j] (H_5 - H_6 + H_7) \right\} \psi_c(p); \end{aligned} \quad (\text{A.14})$$

adding the contribution from Eq. (A.10) to (A.14), we get for the total μ/M correction,

$$\begin{aligned} \Delta T_{ji} &= \phi_j^{(-)} \phi_i^{(+)} \frac{g_u^2}{M} \left(\frac{\partial \alpha'}{\partial \alpha} \right)_{\alpha=0}^2 Z_3 \bar{\psi}_c(p) [\tau_j, \tau_i] \frac{p \cdot q}{4M^2} \\ &\quad \times \{1 + 4M^2(D' - F') - 4H + 2(H_5 - H_6 + H_7)\}. \end{aligned} \quad (\text{A.15})$$

This result is a purely formal one at present, since the unknown functions involved cannot be calculated exactly. It is, therefore, not very useful to compare it with experiment, as could be done via Eqs. (5.2). It may be possible, however, to relate some of these functions to other low energy processes, in which case such a comparison would be of interest.